

# Inertial Effects in Nonequilibrium Work Fluctuations by a Path Integral Approach

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Received: 7 June 2007 / Accepted: 7 August 2007 / Published online: 13 September 2007  
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**Abstract** Inertial effects in fluctuations of the work to sustain a system in a nonequilibrium steady state are discussed for a dragged massive Brownian particle model using a path integral approach. We calculate the work distribution function in the laboratory and comoving frames and prove the asymptotic fluctuation theorem for these works for any initial condition. Important and observable differences between the work fluctuations in the two frames appear for finite times and are discussed concretely for a nonequilibrium steady state initial condition. We also show that for finite times a time oscillatory behavior appears in the work distribution function for masses larger than a nonzero critical value.

**Keywords** Inertial effects · Critical mass · Nonequilibrium work fluctuations and theorem · Path integration · Laboratory and comoving frames

## 1 Introduction

In recent years, fluctuations in nonequilibrium systems have drawn considerable attention to a new kind of fluctuation theorems. These fluctuation theorems are asymmetric relations for the distribution functions for work, heat, etc., and may be satisfied even far from equilibrium states or for small systems in which the magnitude of the fluctuations can be large. These fluctuation theorems have been proved for deterministic thermostated systems [1–4] as well as for stochastic systems [5, 6], and have also been discussed in connection with the Onsager–Machlup fluctuation theory [7]. Moreover, experimental confirmations for these theorems have been obtained [8–11]. It has also been shown that the fluctuation theorems include the fluctuation-dissipation theorem, as well as Onsager’s reciprocal relations, near equilibrium states [1, 6, 12, 13].

In our previous paper [7], based on a generalization of the Onsager–Machlup theory for fluctuations around equilibrium to those around nonequilibrium steady states using a path integral approach, we discussed fluctuation theorems for a stochastic dynamics described

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by a Langevin equation. For a Brownian particle driven by a mechanical force  $F(x_s, s)$ , the Langevin equation for the particle position  $x_s$  at time  $s$  is of the general form

$$m \frac{d^2 x_s}{ds^2} = -\alpha \frac{dx_s}{ds} + F(x_s, s) + \zeta_s \quad (1)$$

with the mass  $m$  of the particle, the friction coefficient  $\alpha$  and a random noise  $\zeta_s$ . In our previous paper, as a nonequilibrium model we considered a dragged Brownian particle, in which the mechanical force is given by a harmonic force  $F(x_s, s) = -\kappa(x_s - v_s)$  with the spring constant  $\kappa$  and the dragging velocity  $v$ . Furthermore we mainly considered this model under the over-damped assumption. This assumption can be used for a dynamics on a much longer time scale than the inertial characteristic time  $\tau_m \equiv m/\alpha$ , and the dynamical equation under this assumption is simply given by neglecting the inertial term containing the mass in (1), i.e. by

$$\frac{dx_s}{ds} = \frac{1}{\alpha} F(x_s, s) + \frac{1}{\alpha} \zeta_s. \quad (2)$$

Equation (2) is much simpler than (1), but information of the system on the shorter time scale than  $\tau_m$  is lost in (2). It may be noted that Machlup and Onsager already developed their fluctuation theory around equilibrium not only for the case corresponding to the over-damped case [14] but also for the inertial case [15]. In our previous paper we discussed also a generalization of the Onsager–Machlup theory for nonequilibrium steady states including the inertial term [7]. However, there we treated only one type of fluctuation theorem, the so called transient fluctuation theorem [2], which is restricted to equilibrium initial conditions. Another fluctuation theorem, the asymptotic fluctuation theorem [3, 4], which holds for any initial condition (including a nonequilibrium steady state<sup>1</sup>), was not discussed for inertial cases in [7]. Different from the transient fluctuation theorem, which is correct for all times as a mathematical identity [16], the asymptotic fluctuation theorem is satisfied in the long time limit only. However, as we will discuss in this paper, a variety of interesting inertial effects appear for finite times for a nonequilibrium initial condition, before the asymptotic fluctuation theorem is achieved. Although there are some results for fluctuation theorems for stochastic systems including inertia [17–19], the asymptotic fluctuation theorem with inertia has not been discussed fully in connection with the Onsager–Machlup theory so far.

The purpose of this paper is therefore to discuss, in the context of the Onsager–Machlup path integral approach, inertial effects in nonequilibrium steady state work fluctuations, including the asymptotic fluctuation theorem. For these discussions we use the Langevin equation (1) for a dragged Brownian particle without the over-damped assumption. The work distribution function is calculated explicitly for any initial condition, and its finite time properties are investigated. As an important inertial effect we show a critical value of the mass above which the work distribution function shows a time-oscillatory behavior.

The nonequilibrium work used in this paper is based on the generalized Onsager–Machlup theory, as obtained in our previous paper [7]. In that paper we considered two kinds of work in two different frames: (A) the work  $\mathcal{W}_l$  done in the laboratory frame ( $l$ ), and (B) the work  $\mathcal{W}_c$  done in the comoving frame ( $c$ ), where the average velocity of the Brownian particle is zero in a nonequilibrium steady state. A difference between these two works

<sup>1</sup>A fluctuation theorem for a nonequilibrium steady state initial condition has been called the steady state fluctuation theorem (or the Gallavotti–Cohen fluctuation theorem [3, 4]), which is a special case of asymptotic fluctuation theorems.

is that  $\mathcal{W}_c$  includes a d’Alembert-like force, which is absent in  $\mathcal{W}_l$ . In this paper, we show that both the works  $\mathcal{W}_l$  and  $\mathcal{W}_c$  satisfy the asymptotic fluctuation theorem. We also discuss dramatic differences between the work distribution functions for  $\mathcal{W}_l$  and  $\mathcal{W}_c$  for finite times.

The outline of this paper is as follows. In Sect. 2 we introduce a dragged Brownian particle model with inertia, and treat its dynamics using a path integral. In Sect. 3 we introduce the works done in the laboratory and comoving frames and calculate their distribution functions. In Sect. 4 we prove the asymptotic work fluctuation theorem. In Sect. 5 we discuss inertial effects in the work distribution functions for finite times. Finally, Sect. 6 is devoted to a summary and some remarks on this paper.

## 2 Dragged Brownian Particle with Inertia

We consider a Brownian particle confined by a harmonic potential, which moves with a constant velocity  $v$  through a fluid, as discussed in our previous paper [7]. The dynamics of this particle is described by a Langevin equation

$$m \frac{d^2 x_s}{ds^2} = -\alpha \frac{dx_s}{ds} - \kappa(x_s - vs) + \zeta_s. \tag{3}$$

Here, we assume that  $\zeta_s$  is the Gaussian-white random force whose probability functional  $P_\zeta(\{\zeta_s\})$  for  $\{\zeta_s\}_{s \in [t_0, t]}$  is given by

$$P_\zeta(\{\zeta_s\}) = C_\zeta \exp\left(-\frac{\beta}{4\alpha} \int_{t_0}^t ds \zeta_s^2\right) \tag{4}$$

with the normalization coefficient  $C_\zeta$  and the inverse temperature  $\beta \equiv 1/(k_B T)$ , where  $k_B$  is Boltzmann’s constant and  $T$  is the temperature of the heat reservoir. [Note that the coefficient  $C_\zeta$  can depend on the initial time  $t_0$  and the final time  $t$ , but such time dependences in  $C_\zeta$ , as well as in similar coefficients  $C_x$  and  $C_\mathcal{E}$  introduced later, are suppressed.] It follows from (4) that the first two auto-correlation functions of the random force  $\zeta_s$  are given by  $\langle \zeta_s \rangle = 0$  and  $\langle \zeta_{s_1} \zeta_{s_2} \rangle = (2\alpha/\beta)\delta(s_1 - s_2)$  with the notation  $\langle \dots \rangle$  for an ensemble average.

Now, we consider the probability functional  $P_x(\{x_s\})$  for a path  $\{x_s\}_{s \in [t_0, t]}$  of the particle position  $x_s$ . By inserting (3) into (4) and interpreting the probability functional  $P_\zeta(\{\zeta_s\})$  for  $\zeta_s$  as the probability functional  $P_x(\{x_s\})$  for  $x_s$ , we obtain, apart from a normalization coefficient,

$$P_x(\{x_s\}) = C_x \exp\left[-\frac{1}{4D} \int_{t_0}^t ds \left(\dot{x}_s + \frac{x_s - vs}{\tau_r} + \frac{m}{\alpha} \ddot{x}_s\right)^2\right] \tag{5}$$

with  $\dot{x}_s \equiv dx_s/ds$ ,  $\ddot{x}_s \equiv d^2x_s/ds^2$  and the normalization coefficient  $C_x$ . Here,  $D \equiv k_B T/\alpha$  is the diffusion constant given by the Einstein relation and  $\tau_r \equiv \alpha/\kappa$  is the relaxation time in the over-damped case. For another derivation of (5) via a Fokker–Planck equation corresponding to the Langevin equation, see, for example, [20].

For systems whose dynamics is expressed by a second-order Langevin equation, like (3), we introduce the path integration of any functional  $X(\{x_s\})$  as  $\int_{(x_{t_0}, \dot{x}_{t_0})=(x_i, p_i/m)}^{(x_t, \dot{x}_t)=(x_f, p_f/m)} \mathcal{D}x_s X(\{x_s\})$ , with respect to paths  $\{x_s\}_{s \in [t_0, t]}$  satisfying the initial ( $i$ ) condition  $(x_{t_0}, \dot{x}_{t_0}) = (x_i, p_i/m)$  and the final ( $f$ ) condition  $(x_t, \dot{x}_t) = (x_f, p_f/m)$ . Using this notation for the functional integral,

the functional average  $\langle\langle X(\{x_s\})\rangle\rangle_t$  over all possible paths  $\{x_s\}_{s \in (t_0, t)}$ , as well as averages over the initial and final positions and momenta of the particle is represented by

$$\begin{aligned} \langle\langle X(\{x_s\})\rangle\rangle_t &\equiv \iint dx_i dp_i \int_{(x_{t_0}, \dot{x}_{t_0})=(x_i, p_i/m)}^{(x_t, \dot{x}_t)=(x_f, p_f/m)} \mathcal{D}x_s \iint dx_f dp_f \\ &\times X(\{x_s\}) P_x(\{x_s\}) f(x_i, p_i, t_0) \end{aligned} \quad (6)$$

with the initial distribution function  $f(x_i, p_i, t_0)$  for the particle position  $x_i$  and momentum  $p_i$ . The normalization condition to specify the coefficient  $C_x$  of the distribution functional (5) is given by  $\langle\langle 1 \rangle\rangle_t = 1$  using the notation (6) as well as the normalization condition  $\iint dx_i dp_i f(x_i, p_i, t_0) = 1$  for the initial distribution function.

This finishes the introduction of our model and its dynamics. In Sect. 3 we introduce the work done on this system and calculate its probability distribution.

### 3 Work Distribution

#### 3.1 Work to Drag a Brownian Particle and its Distribution

In our previous paper [7], we considered the work  $\mathcal{W}$  to move the confining potential with a velocity  $v$  in two frames; the laboratory frame using the particle position  $x_s$  and the comoving frame using the particle position  $y_s \equiv x_s - vs$  at time  $s$ . Based on a generalized Onsager–Machlup theory, we showed in [7] that the work  $\mathcal{W}_l$  done in the laboratory frame is given by  $\int_{t_0}^t ds [-\kappa(x_s - vs)]v$ , and the work  $\mathcal{W}_c$  done in the comoving frame is given by  $\int_{t_0}^t ds (-\kappa y_s - m\ddot{y}_s)v$  with  $\ddot{y}_s \equiv d^2 y_s / ds^2 = \ddot{x}_s$ , leading to a difference between the work  $\mathcal{W}$  in these two frames by an inertial or d’Alembert-like force  $-m\ddot{y}_s$ . To understand this difference in a concise way, note first that by the energy conservation law, the work  $\mathcal{W}$  is given by the heat  $Q$  and the energy difference  $\Delta E$ , namely by  $\mathcal{W} = Q + \Delta E$ , where the energy difference  $\Delta E$  is the sum of the kinetic energy difference  $\Delta K$  and the potential energy difference  $\Delta U$ , i.e.  $\Delta E = \Delta K + \Delta U$ . Here, the kinetic energy difference  $\Delta K$  is either  $\Delta K_c$  or  $\Delta K_l$  in the comoving frame or the laboratory frame, i.e. given either by  $(m\dot{y}_t^2/2) - (m\dot{y}_{t_0}^2/2)$  or by  $(m\dot{x}_t^2/2) - (m\dot{x}_{t_0}^2/2)$ , respectively, so that we obtain the relation

$$\Delta K_c = \Delta K_l - \int_{t_0}^t ds m\ddot{x}_s v. \quad (7)$$

Equation (7) means that the kinetic energy difference  $\Delta K$  depends on the frames and its frame-difference is determined by the d’Alembert-like force  $-m\ddot{x}_s$  as a purely inertial effect. On the other hand, different from  $\Delta K$ , the heat  $Q$  (as well as the potential energy difference  $\Delta U$ ) has the same value in the laboratory and the comoving frames.<sup>2</sup> Combining these facts with the energy conservation law  $\mathcal{W} = Q + \Delta K + \Delta U$ , we obtain the work relation  $\mathcal{W}_c = \mathcal{W}_l - \int_{t_0}^t ds m\ddot{x}_s v$ . A more complete explanation for this frame-dependence of the work is given in [7].<sup>3</sup>

<sup>2</sup>This follows from the generalized Onsager–Machlup theory in which the heat  $Q$  is defined by  $Q = T \int_{t_0}^t ds \dot{S}$  with the entropy production rate  $\dot{S}$  derived from a Lagrangian function [cf. (18) and (21) in [7]].

<sup>3</sup>In [7], using a nonequilibrium generalization of the detailed balance condition [cf. (97) and (98) in [7]] we showed that the force difference between the comoving and laboratory frame is given by the d’Alembert-like force  $-m\ddot{x}_s$  [cf. Table 1 in [7]], implying a work difference  $-\int_{t_0}^t ds m\ddot{x}_s v$  in these two frames.

To discuss these two different kinds of work done in the laboratory and comoving frames simultaneously in this paper, we consider the work defined in general by

$$\mathcal{W}(\{x_s\}) = \int_{t_0}^t ds [-\kappa(x_s - vs) - (1 - \vartheta)m\ddot{x}_s]v, \tag{8}$$

which gives the work  $\mathcal{W}_l$  done in the laboratory case ( $\vartheta = 1$ ) as well as the work  $\mathcal{W}_c$  done in the comoving case ( $\vartheta = 0$ ) by changing value of the parameter  $\vartheta$ .<sup>4</sup>

Using the functional average defined by (6), the probability distribution  $P_w(W)$  for the dimensionless work  $\beta\mathcal{W}(\{x_s\})$  is given by

$$P_w(W, t) = \langle\langle \delta(W - \beta\mathcal{W}(\{x_s\})) \rangle\rangle_t. \tag{9}$$

For later calculative convenience, we introduce a Fourier transformation  $\mathcal{E}_w(i\lambda, t)$  of the work distribution function  $P_w(W, t)$  through the function  $\mathcal{E}_w(\lambda, t)$  defined by

$$\mathcal{E}_w(\lambda, t) \equiv \langle\langle e^{-\lambda\beta\mathcal{W}(\{x_s\})} \rangle\rangle_t, \tag{10}$$

so that the work distribution function  $P_w(W)$  can be represented as

$$P_w(W, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda \mathcal{E}_w(i\lambda, t) e^{i\lambda W}. \tag{11}$$

The function  $\mathcal{E}_w(\lambda, t)$  can be also regarded as a generating function for the work  $\mathcal{W}(\{x_s\})$ . By (10) we obtain a useful identity

$$\mathcal{E}_w(0, t) = 1 \tag{12}$$

used to determine a normalization constant later (43).

### 3.2 Path Integral Analysis for Work Distribution

To calculate the function  $\mathcal{E}_w(\lambda, t)$  from (10), we first note that

$$\begin{aligned} \mathcal{E}_w(\lambda, t) = C_x & \iint dx_i dp_i \int_{(x_{t_0}, \dot{x}_{t_0})=(x_i, p_i/m}^{(x_t, \dot{x}_t)=(x_f, p_f/m)} \mathcal{D}x_s \iint dx_f dp_f \\ & \times f(x_i, p_i, t_0) \exp \left[ \int_{t_0}^t ds L(\ddot{x}_s, \dot{x}_s, x_s, s) \right] \end{aligned} \tag{13}$$

by (5, 6, 8, 10). Here,  $L(\ddot{x}_s, \dot{x}_s, x_s, s)$  is defined by

$$\begin{aligned} L(\ddot{x}_s, \dot{x}_s, x_s, s) \equiv & -\frac{1}{4D} \left( \dot{x}_s + \frac{x_s - vs}{\tau_r} + \frac{m}{\alpha} \ddot{x}_s \right)^2 \\ & + \lambda\beta[\kappa(x_s - vs) + (1 - \vartheta)m\ddot{x}_s]v, \end{aligned} \tag{14}$$

which may be interpreted as a Lagrangian function including a Lagrange multiplier  $\lambda$  due to the restriction of the delta function for work in (9) [7].<sup>5</sup> Here, as elsewhere in this paper, the

<sup>4</sup>The parameter  $\vartheta$  in (8) is chosen in a way consistent to that in our previous paper [7].

<sup>5</sup>In [7] we called only the first term on the right-hand side of (14) the Lagrangian function in the Onsager–Machlup theory, which is directly connected to a transition probability.

dependence of  $L(\ddot{x}_s, \dot{x}_s, x_s, s)$  on the parameters  $v, \vartheta$ , etc., has not been explicitly indicated on the left-hand side of (14).

The first step to calculate the function  $\mathcal{E}_w(\lambda, t)$  is to specify the most-contributing path  $\{x_s^*\}_{s \in [t_0, t]}$  in the path integral involved on the right-hand side of (13). Such a special path  $\{x_s^*\}_{s \in [t_0, t]}$  is introduced as the one satisfying the variational principle

$$\delta \int_{t_0}^t ds L(\ddot{x}_s^*, \dot{x}_s^*, x_s^*, s) = 0 \tag{15}$$

with the four boundary conditions  $x_{t_0}^* = x_i, \dot{x}_{t_0}^* = p_i/m, x_t^* = x_f$  and  $\dot{x}_t^* = p_f/m$ . In a way similar to derive the Euler–Lagrange equation from the minimum action principle in analytical mechanics [21], (15) leads to

$$\frac{d^2}{ds^2} \frac{\partial L(\ddot{x}_s^*, \dot{x}_s^*, x_s^*, s)}{\partial \ddot{x}_s^*} - \frac{d}{ds} \frac{\partial L(\ddot{x}_s^*, \dot{x}_s^*, x_s^*, s)}{\partial \dot{x}_s^*} + \frac{\partial L(\ddot{x}_s^*, \dot{x}_s^*, x_s^*, s)}{\partial x_s^*} = 0. \tag{16}$$

Inserting (14) into (16) we obtain a fourth-order linear differential equation

$$\tau_m^2 \frac{d^4 \tilde{x}_s^*}{ds^4} - \left(1 - 2 \frac{\tau_m}{\tau_r}\right) \frac{d^2 \tilde{x}_s^*}{ds^2} + \frac{1}{\tau_r^2} \tilde{x}_s^* = 0 \tag{17}$$

for the function  $\tilde{x}_s^*$  of  $s$ , which is defined by

$$\tilde{x}_s^* \equiv x_s^* - vs + (1 - 2\lambda)v\tau_r, \tag{18}$$

using the inertial characteristic time  $\tau_m \equiv m/\alpha$ .

We consider solutions of (17) of the form  $\exp(vs)$ . Inserting  $\tilde{x}_s^* = \exp(vs)$  into (17) we obtain the quadratic equation

$$\tau_m^2 v^4 - \left(1 - 2 \frac{\tau_m}{\tau_r}\right) v^2 + \frac{1}{\tau_r^2} = \left(\tau_m v^2 + v + \frac{1}{\tau_r}\right) \left(\tau_m v^2 - v + \frac{1}{\tau_r}\right) = 0 \tag{19}$$

for  $v$ . The solutions of (19) are  $v = v_+, v_-, -v_-, -v_+$  using  $v_{\pm}$  defined by

$$v_{\pm} \equiv \frac{1}{2\tau_m} \left(1 \pm \sqrt{1 - 4 \frac{\tau_m}{\tau_r}}\right). \tag{20}$$

The general solution of the fourth-order differential equation (17) is represented as a superposition of these special solutions  $\exp(vs)$ ,  $v = v_+, v_-, -v_-, -v_+$ , namely

$$\tilde{x}_s^* = C_1 e^{v_+ s} + C_2 e^{v_- s} + C_3 e^{-v_- s} + C_4 e^{-v_+ s} \tag{21}$$

with constants  $C_j, j = 1, 2, 3, 4$ . Using (18) and (21) and introducing the 4-dimensional vector  $\mathbf{C} \equiv (C_1 C_2 C_3 C_4)^T$ ,<sup>6</sup> we can rewrite

$$x_s^* = \mathbf{C}^T \mathbf{K}_s + vs - (1 - 2\lambda)v\tau_r \tag{22}$$

<sup>6</sup>In this paper,  $X^T$  means the transposed matrix (or vector) of any matrix (or vector)  $X$ .

where the 4-dimensional vector  $\mathbf{K}_s$  is defined by

$$\mathbf{K}_s \equiv \begin{pmatrix} e^{\nu_+ s} \\ e^{\nu_- s} \\ e^{-\nu_- s} \\ e^{-\nu_+ s} \end{pmatrix}. \tag{23}$$

The constant vector  $\mathbf{C}$  is determined by the four boundary conditions for  $x_s^*$  and we obtain

$$\mathbf{C} = A_t^{-1} \mathbf{B}_{if}^{(1-2\lambda)} \tag{24}$$

where the  $4 \times 4$  matrix  $A_t$  is defined by

$$A_t \equiv \begin{pmatrix} e^{\nu_+ t_0} & e^{\nu_- t_0} & e^{-\nu_- t_0} & e^{-\nu_+ t_0} \\ \nu_+ e^{\nu_+ t_0} & \nu_- e^{\nu_- t_0} & -\nu_- e^{-\nu_- t_0} & -\nu_+ e^{-\nu_+ t_0} \\ e^{\nu_+ t} & e^{\nu_- t} & e^{-\nu_- t} & e^{-\nu_+ t} \\ \nu_+ e^{\nu_+ t} & \nu_- e^{\nu_- t} & -\nu_- e^{-\nu_- t} & -\nu_+ e^{-\nu_+ t} \end{pmatrix} \tag{25}$$

and the 4-dimensional vector  $\mathbf{B}_{if}^{(z)}$  is defined by

$$\mathbf{B}_{if}^{(z)} \equiv \begin{pmatrix} x_i - vt_0 \\ p_i/m - v \\ x_f - vt \\ p_f/m - v \end{pmatrix} + z v \tau_r \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{26}$$

It may be noted that the first component  $x_i - vt_0$  and the second component  $(p_i/m) - v$  [the third component  $x_f - vt_0$  and the fourth component  $(p_f/m) - v$ ] of the vector  $\mathbf{B}_{if}^{(0)}$  can be regarded as the initial [final] position and velocity of the particle in the comoving frame, respectively.

As the next step, we represent a path  $\{x_s\}_{s \in [t_0, t]}$  as the sum of the most contributing path  $\{x_s^*\}_{s \in [t_0, t]}$  given by (22) and its deviation  $\{\Delta x_s\}_{s \in [t_0, t]}$  defined by

$$\Delta x_s \equiv x_s - x_s^*, \tag{27}$$

where the variable  $\Delta x_s$  satisfies the four boundary conditions  $\Delta x_{t_0} = \Delta x_t = 0$  and  $\Delta \dot{x}_{t_0} = \Delta \dot{x}_t = 0$  with  $\Delta \dot{x}_s \equiv d\Delta x_s/ds$ . Using this variable  $\Delta x_s$ , the complete time integral  $\int_{t_0}^t ds L(\ddot{x}_s, \dot{x}_s, x_s, s)$  of  $L(\ddot{x}_s, \dot{x}_s, x_s, s)$  can be represented as

$$\begin{aligned} \int_{t_0}^t ds L(\ddot{x}_s, \dot{x}_s, x_s, s) &= \int_{t_0}^t ds L(\ddot{x}_s^*, \dot{x}_s^*, x_s^*, s) \\ &\quad - \frac{1}{4D} \int_{t_0}^t ds \left( \Delta \dot{x}_s + \frac{1}{\tau_r} \Delta x_s + \frac{m}{\alpha} \Delta \ddot{x}_s \right)^2, \end{aligned} \tag{28}$$

in terms of the two variables  $x_s^*$  and  $\Delta x_s$ . Inserting (28) into (13) we obtain

$$\begin{aligned} \mathcal{E}_w(\lambda, t) &= C_{\mathcal{E}} \iint dx_i dp_i \iint dx_f dp_f \\ &\quad \times f(x_i, p_i, t_0) \exp \left[ \int_{t_0}^t ds L(\ddot{x}_s^*, \dot{x}_s^*, x_s^*, s) \right] \end{aligned} \tag{29}$$

where  $C_\varepsilon$  is defined by  $C_\varepsilon \equiv C_x \int_{(\Delta x_{t_0}, \Delta \dot{x}_{t_0})=(0,0)}^{(\Delta x_t, \Delta \dot{x}_t)=(0,0)} \mathcal{D}\Delta x_s \exp[-(1/4D) \int_{t_0}^t ds \int_{t_0}^t ds (\Delta \dot{x}_s + (1/\tau_r)\Delta x_s + (m/\alpha)\Delta \ddot{x}_s)^2]$  and is independent of  $\lambda$ . In the expression (29), the contributions of the deviations  $\Delta x_s$  to the path integral in the function  $\mathcal{E}_w(\lambda, t)$  are included only in the coefficient  $C_\varepsilon$ .

Next, we calculate the quantity  $\int_{t_0}^t ds L(\ddot{x}_s^*, \dot{x}_s^*, x_s^*, s)$  using (22), and then the function  $\mathcal{E}_w(\lambda, t)$  given by (29). For such a calculation, using (22) we first note that

$$\frac{dx_s^*}{ds} = \mathbf{C}^T \Theta \mathbf{K}_s + v, \tag{30}$$

$$\frac{d^2x_s^*}{ds^2} = \mathbf{C}^T \Theta^2 \mathbf{K}_s, \tag{31}$$

where the  $4 \times 4$  matrix  $\Theta$  is defined by

$$\Theta \equiv \begin{pmatrix} v_+ & 0 & 0 & 0 \\ 0 & v_- & 0 & 0 \\ 0 & 0 & -v_- & 0 \\ 0 & 0 & 0 & -v_+ \end{pmatrix}. \tag{32}$$

Then, using (22), (30) and (31) we obtain

$$\dot{x}_s^* + \frac{x_s^* - vs}{\tau_r} + \frac{m}{\alpha} \ddot{x}_s^* = \mathbf{C}^T \Gamma \mathbf{K}_s + 2\lambda v \tag{33}$$

where the  $4 \times 4$  matrix  $\Gamma$  is introduced as

$$\begin{aligned} \Gamma &\equiv \tau_m \Theta^2 + \Theta + \frac{1}{\tau_r} \mathcal{I} \\ &= 2 \begin{pmatrix} v_+ & 0 & 0 & 0 \\ 0 & v_- & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{34}$$

with the relation  $\tau_m v_\pm^2 - v_\pm + \tau_r^{-1} = 0$  and  $\mathcal{I}$  the  $4 \times 4$  identity matrix. Using (14, 22, 31, 33) we obtain

$$\begin{aligned} L(\ddot{x}_s^*, \dot{x}_s^*, x_s^*, s) &= -\frac{1}{4} \alpha \beta \mathbf{C}^T \Gamma \mathbf{K}_s \mathbf{K}_s^T \Gamma \mathbf{C} - \alpha \beta \lambda v \mathbf{C}^T (\tau_m \vartheta \Theta^2 + \Theta) \mathbf{K}_s \\ &\quad - \lambda (1 - \lambda) \alpha \beta v^2. \end{aligned} \tag{35}$$

Noting (24) and that

$$\int_{t_0}^t ds \mathbf{C}^T \Theta \mathbf{K}_s = \int_{t_0}^t ds \frac{dx_s^*}{ds} - v(t - t_0) = x_f - x_i - v(t - t_0), \tag{36}$$

$$\int_{t_0}^t ds \mathbf{C}^T \Theta^2 \mathbf{K}_s = \int_{t_0}^t ds \frac{d^2x_s^*}{ds^2} = \frac{p_f - p_i}{m} \tag{37}$$

by (30) and (31), we further obtain

$$\int_{t_0}^t ds L(\ddot{x}_s^*, \dot{x}_s^*, x_s^*, s) = -\frac{1}{4} \alpha \beta [\mathbf{B}_{if}^{(1-2\lambda)}]^T \Lambda_t \mathbf{B}_{if}^{(1-2\lambda)} - \alpha \beta \lambda v \boldsymbol{\eta}^T \mathbf{B}_{if}^{(0)}$$



$$-\lambda(1 - \lambda)\alpha\beta v^2(t - t_0) \tag{38}$$

where the  $4 \times 4$  matrix  $\Lambda_t$  and the 4-dimensional vector  $\boldsymbol{\eta}$  are defined by

$$\Lambda_t \equiv (A_t^{-1})^T \Gamma \Phi_t \Gamma A_t^{-1}, \tag{39}$$

$$\boldsymbol{\eta} \equiv \begin{pmatrix} -1 \\ -\tau_m \vartheta \\ 1 \\ \tau_m \vartheta \end{pmatrix}, \tag{40}$$

respectively, with the  $4 \times 4$  matrix  $\Phi_t$  defined by

$$\Phi_t \equiv \int_{t_0}^t ds \mathbf{K}_s \mathbf{K}_s^T. \tag{41}$$

Inserting (38) into (29) we obtain

$$\begin{aligned} \mathcal{E}_w(\lambda, t) &= C_\mathcal{E} e^{-\lambda(1-\lambda)\alpha\beta v^2(t-t_0)} \iint dx_i dp_i \iint dx_f dp_f f(x_i, p_i, t_0) \\ &\times \exp \left\{ -\frac{1}{4} \alpha \beta [\mathbf{B}_{if}^{(1-2\lambda)}]^T \Lambda_t \mathbf{B}_{if}^{(1-2\lambda)} - \alpha \beta \lambda v \boldsymbol{\eta}^T \mathbf{B}_{if}^{(0)} \right\}. \end{aligned} \tag{42}$$

Equation (42) gives a concrete form of the function  $\mathcal{E}_w(\lambda, t)$  for any initial distribution function  $f(x_i, p_i, t_0)$ .

The  $\lambda$ -independent normalization coefficient  $C_\mathcal{E}$  in (42) can be determined from the condition (12), and we obtain

$$\begin{aligned} C_\mathcal{E} &= \left\{ \iint dx_i dp_i \iint dx_f dp_f f(x_i, p_i, t_0) \right. \\ &\times \left. \exp \left\{ -\frac{1}{4} \alpha \beta [\mathbf{B}_{if}^{(1)}]^T \Lambda_t \mathbf{B}_{if}^{(1)} \right\} \right\}^{-1}. \end{aligned} \tag{43}$$

Note that by using the condition (12) we avoided to carry out explicitly the path integral included originally in the quantity  $C_\mathcal{E}$  [cf. (29)].

Inserting (42) into (11), and carrying out the Gaussian integral over  $\lambda$  appearing then in (11), we obtain

$$\begin{aligned} P_w(W, t) &= \frac{C_\mathcal{E}}{\sqrt{4\pi\alpha\beta v^2(t-t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})}} \iint dx_i dp_i \iint dx_f dp_f \\ &\times f(x_i, p_i, t_0) \exp \left\{ -\frac{1}{4} \alpha \beta [\mathbf{B}_{if}^{(1)}]^T \Lambda_t \mathbf{B}_{if}^{(1)} \right\} \\ &\times \exp \left\{ -\frac{\{W - \alpha\beta v[v(t-t_0) + (\boldsymbol{\eta}^T - \tau_r \mathbf{J}^T \Lambda_t) \mathbf{B}_{if}^{(1)}]\}^2}{4\alpha\beta v^2(t-t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})} \right\} \end{aligned} \tag{44}$$

where the 4-dimensional vector  $\mathbf{J}$  is defined by

$$\mathbf{J} \equiv \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{45}$$

and we used the relation  $\boldsymbol{\eta}^T \mathbf{J} = 0$ . Equation (44) is an explicit form for the work distribution function for all time, and for any initial distribution function  $f(x_i, p_i, t_0)$ . Using (43) for the coefficient  $C_{\mathcal{E}}$ , the work distribution function (44) is properly normalized, namely  $\int dW P_w(W, t) = 1$ , at any time  $t$ .

In Sects. 4 and 5 we discuss, using the work distribution function (44), fluctuation properties of the work from the viewpoint of the asymptotic fluctuation theorem for  $t \rightarrow +\infty$ , as well as for finite times.

### 4 Asymptotic Fluctuation Theorem

The matrix  $\Lambda_t$  defined by (39) satisfies the condition

$$\lim_{t \rightarrow +\infty} \frac{1}{t - t_0} \Lambda_t = 0, \tag{46}$$

as shown in Appendix 1. Equation (46) implies that  $v(t - t_0) + (\boldsymbol{\eta}^T - \tau_r \mathbf{J}^T \Lambda_t) \mathbf{B}_{if}^{(1)} \overset{t \rightarrow +\infty}{\sim} v(t - t_0)$  and  $t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J} \overset{t \rightarrow +\infty}{\sim} t - t_0$  in (44), so that the work distribution function  $P_w(W, t)$  is proportional to the Gaussian function  $\exp\{-[W - \alpha\beta v^2(t - t_0)]^2/[4\alpha\beta v^2(t - t_0)]\}$  in the long time limit  $t \rightarrow +\infty$ , i.e.

$$P_w(W, t) \overset{t \rightarrow +\infty}{\sim} \frac{1}{\sqrt{4\pi\alpha\beta v^2(t - t_0)}} \exp\left\{-\frac{[W - \alpha\beta v^2(t - t_0)]^2}{4\alpha\beta v^2(t - t_0)}\right\} \tag{47}$$

regardless of the initial distribution function  $f(x_i, p_i, t_0)$ . It is important to note that the work distribution function (47) in the long time limit  $t \rightarrow +\infty$  in the inertial case is the same as in the over-damped case. Physically, this is, of course, due to the finiteness of the inertial characteristic time  $\tau_m$ , which makes inertial effects disappear in the long time limit. Nevertheless, the proof of this equivalence is non-trivial.

From (47) we immediately derive

$$\lim_{t \rightarrow +\infty} \frac{P_w(W, t)}{P_w(-W, t)} = e^W \tag{48}$$

for any initial distribution function  $f(x_i, p_i, t_0)$ . We will call (48) the asymptotic fluctuation theorem for work. Equation (48) is independent of the value of the parameter  $\vartheta$ , i.e. of the frame of reference (laboratory or comoving) or also of the contribution of the d'Alembert-like force to the work (8).

### 5 Inertial Effects for Finite Times

#### 5.1 Slope of $\ln[P_w(W, t)/P_w(-W, t)]$ and the Critical Mass

In contrast to the asymptotic work distribution function (47), various inertial effects in the work distribution function appear for finite times. In this section we discuss such inertial

effects using the function  $G(W, t)$  defined by

$$G(W, t) \equiv \frac{\partial}{\partial W} \ln \frac{P_w(W, t)}{P_w(-W, t)}. \tag{49}$$

The function  $G(W, t)$  gives the slope of the fluctuation function  $\ln[P_w(W, t) / P_w(-W, t)]$  with respect to  $W$ , and satisfies

$$\lim_{t \rightarrow +\infty} G(W, t) = 1 \tag{50}$$

by the asymptotic fluctuation theorem (48).<sup>7</sup>

The behavior of  $G(W, t)$  for finite times depends on the initial condition. To get concrete results, in this section we concentrate on the case of a nonequilibrium steady state initial condition, which can be represented by

$$f(x_i, p_i, t_0) = \frac{\beta}{2\pi} \sqrt{\frac{\kappa}{m}} \exp \left\{ -\beta \left[ \frac{(p_i - mv)^2}{2m} + \frac{1}{2} \kappa (x_i - vt_0 + v\tau_r)^2 \right] \right\} \tag{51}$$

for any frame.<sup>8</sup> The initial distribution function (51) gives a Gaussian distribution for the particle initial position  $x_i$  and momentum  $p_i$  around their nonequilibrium steady state average values  $vt_0 - v\tau_r$  and  $mv$ , respectively. Inserting (51) into (44) the work distribution function is given by

$$P_w(W, t) = \sqrt{\frac{1 - \Omega_t}{4\pi\alpha\beta v^2(t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})}} \times \exp \left\{ -\frac{1 - \Omega_t}{4\alpha\beta v^2(t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})} [W - \alpha\beta v^2(t - t_0)]^2 \right\} \tag{52}$$

where  $\Omega_t$  is defined by

$$\Omega_t \equiv (\boldsymbol{\eta} - \tau_r \Lambda_t \mathbf{J})^T [(\boldsymbol{\eta} - \tau_r \Lambda_t \mathbf{J})(\boldsymbol{\eta} - \tau_r \Lambda_t \mathbf{J})^T + (t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})(\Lambda^{(0)} + \Lambda_t)]^{-1} (\boldsymbol{\eta} - \tau_r \Lambda_t \mathbf{J}) \tag{53}$$

with the  $4 \times 4$  matrix  $\Lambda^{(0)}$  defined by

$$\Lambda^{(0)} \equiv \frac{2}{\alpha} \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{54}$$

<sup>7</sup>A function like  $\tilde{G}(W, t) \equiv (1/\langle W \rangle) \ln[P_w(W, t)/P_w(-W, t)]$  with the average work  $\langle W \rangle$  has been used to characterize fluctuation theorems [22, 23]. The function (49) is connected to  $\tilde{G}(W, t)$  by  $G(W, t) = \langle W \rangle \partial \tilde{G}(W, t) / \partial W$ . One of the advantage to use  $G(W, t)$  instead of  $\tilde{G}(W, t)$  is that different from  $\tilde{G}(W, t)$ ,  $G(W, t)$  is independent of  $W$  when the distribution function  $P_w(W, t)$  is Gaussian, as shown in (52).

<sup>8</sup>To show that the distribution (51) is a nonequilibrium steady state distribution, we note that from (3) the Langevin equation for the variable  $\tilde{y}_s \equiv x_s - vs + v\tau_r$  (so that  $\tilde{p}_s \equiv m d\tilde{y}_s/ds \equiv m\dot{x}_s - mv$ ) is given by  $m d^2\tilde{y}_s/ds^2 = -\alpha d\tilde{y}_s/ds - \kappa \tilde{y}_s + \zeta_s$ , which has the same form as that for equilibrium. Therefore, the equilibrium canonical distribution  $[\beta/(2\pi)] \sqrt{\kappa/m} \exp\{-\beta[\tilde{p}_s^2/(2m) + \frac{1}{2}\kappa \tilde{y}_s^2]\}$  is a steady state solution of the corresponding Kramers equation [20]. Rewriting this steady state distribution using  $x_s$  instead of  $\tilde{y}_s$  we obtain (51).

[See Appendix 2 for a derivation of (52).] Note that the work distribution function (52) is Gaussian with the average work  $\langle W \rangle = \alpha\beta v^2(t - t_0)$  at any time because we chose a Gaussian nonequilibrium steady state initial condition (51). Since the work distribution function  $P(W, t)$  is Gaussian,  $G(W, t)$  defined by (49) is independent of  $W$ , so that we denote it by  $G(t)$  [=  $G(W, t)$ ] from now on. Inserting (52) into (49), we obtain

$$G(t) = \frac{1 - \Omega_t}{1 - \frac{\tau_r^2}{t-t_0} \mathbf{J}^T \Lambda_t \mathbf{J}} \tag{55}$$

as an explicit form of  $G(t)$ . One may notice that  $G(t)$  in (55) is independent of the dragging velocity  $v$  and the inverse temperature  $\beta$ . Moreover,  $G(t)$  is positive for  $t > t_0$  because the distribution function  $P_w(W, t)$  is normalizable so that the coefficient  $(1 - \Omega_t)/[4\alpha\beta v^2(t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})] = G(t)/[4\alpha\beta v^2(t - t_0)]$  in the exponent of the Gaussian distribution function (52) must be positive.

As a first approximation to the asymptotic relaxation of  $G(t)$  to its final value (50), we obtain from (55)

$$G(t) \stackrel{t \rightarrow +\infty}{\sim} 1 + \frac{\tau_r - \tau_m \vartheta^2}{t - t_0 - \tau_r + \tau_m \vartheta^2}, \tag{56}$$

meaning that the function  $G(t)$  decays to 1 by a power inversely proportional to the time in the long time limit  $t \rightarrow +\infty$ . [See Appendix 3 for a derivation of (56).] Equation (56) is only the first approximation for an asymptotic form of  $G(t)$ , but already includes an important inertial contribution to  $G(t)$ , as well as an interesting frame dependence of  $G(t)$ . Actually, the second term on the right-hand side of (56) depends on the mass  $m$  via  $\tau_m = m/\alpha$  in the laboratory frame  $\vartheta = 1$ , while that term is independent of the mass in the comoving frame  $\vartheta = 0$ . Another interesting property of  $G(t)$  expressed by (56) is that in the laboratory frame  $\vartheta = 1$  the second term on the right-hand side of (56), the  $t^{-1}$ -decay term of  $G(t)$ , vanishes in the case that  $\tau_r = \tau_m$ , i.e. for a special mass value  $m = \alpha^2/\kappa$ .

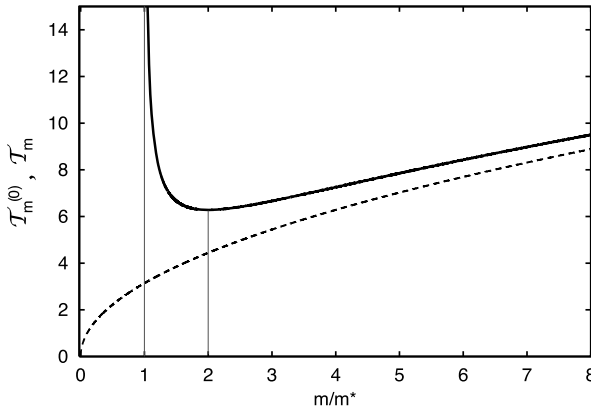
Perhaps the most interesting implication of (55) for  $G(t)$ , although it does not appear explicitly in the asymptotic expression (56) of  $G(t)$ , is the existence of a critical value of the mass  $m = m^*$  above which  $G(t)$  shows a time-oscillation behavior. In our theory, this time-oscillation has its origin in the time-dependence of  $\tilde{x}_s^*$  given by (21) via the exponential terms  $\exp(v_{\pm}t)$ , etc., when the coefficient  $v_{\pm}$  given by (20) has an imaginary part, namely when the condition

$$m > m^* \equiv \frac{\alpha^2}{4\kappa} \tag{57}$$

(derived from the condition  $4\tau_m \tau_r^{-1} - 1 > 0$ ) is satisfied. We call the mass  $m^*$  the critical mass in this paper, since a (smooth) “dynamical” phase transition takes place at  $m = m^*$ . For masses  $m > m^*$ , the position  $\tilde{x}_s^*$  has a time-oscillation with the oscillation period  $\mathcal{T}_m$

$$\mathcal{T}_m = 2\pi \sqrt{\frac{m}{\kappa} \left(1 - \frac{m^*}{m}\right)^{-1/2}} \tag{58}$$

corresponding to a frequency  $\omega \equiv \sqrt{4\tau_m \tau_r^{-1} - 1}/(2\tau_m) = |\text{Im}\{v_{\pm}\}|$ , using the imaginary part  $\text{Im}\{v_{\pm}\}$  of  $v_{\pm}$ . In Fig. 1 the time-oscillation period  $\mathcal{T}_m$  (solid line) is shown as a function of the scaled mass  $m/m^*$  for  $m/m^* > 1$ . There is no time-oscillation of  $\tilde{x}_s^*$  in the case of  $m/m^* < 1$ , and the time-oscillation period diverges when  $m/m^* \rightarrow 1 + 0$ . The oscillation period  $\mathcal{T}_m$



**Fig. 1** Time oscillation period  $\mathcal{T}_m$  (solid line) in  $\bar{x}_s^*$  as a function of mass  $m$  normalized by the critical mass  $m^* \equiv \alpha^2/(4\kappa)$  for the mass in  $m/m^* > 1$ . There is no time-oscillation for  $m/m^* < 1$  and the minimum of  $\mathcal{T}_m$  is at  $m/m^* = 2$ . Here, we used parameter values  $\alpha = \kappa = 1$ . We also plotted a time-oscillation period  $\mathcal{T}_m^{(0)} = \mathcal{T}_m|_{\alpha=0} (= 2\pi\sqrt{m/\kappa})$  (broken line) for a purely harmonic oscillator in the case without dissipation ( $\alpha = 0$ ). The time-oscillation period  $\mathcal{T}_m$  approaches  $\mathcal{T}_m^{(0)}$  in the large mass limit  $m/m^* \rightarrow +\infty$

decreases rapidly as a function of mass  $m$  for  $m/m^* < 2$ , has a minimum at  $m/m^* = 2$ , and increases gradually for  $m/m^* > 2$ . For comparison, we also plotted in Fig. 1 the scaled mass dependence of the time-oscillation period  $\mathcal{T}_m^{(0)} = 2\pi\sqrt{m/\kappa} (= \mathcal{T}_m|_{\alpha=0})$  (broken line) for a purely harmonic oscillator with spring constant  $\kappa$ . Different from the time-oscillation period (58), the period  $\mathcal{T}_m^{(0)}$  is defined for all masses, and increases monotonically as the mass increases. The time-oscillation period (58) approaches  $\mathcal{T}_m^{(0)}$  in the large mass limit  $m/m^* \rightarrow +\infty$ .

It is useful to consider the critical behavior in the time-oscillating behavior of  $G(t)$  as due to the presence of two independent time scales appearing in our model: one characterized by  $\tau_r (= \alpha/\kappa)$  and another by  $\tau_m (= m/\alpha)$ . These time scales  $\tau_m$  and  $\tau_r$  are related by  $\tau_r = 4\tau_m^*$  at the critical mass  $m = m^*$ . Using these two time scales, the time oscillation period  $\mathcal{T}_m^{(0)}$  for a purely harmonic oscillator is given by  $\mathcal{T}_m^{(0)} = 2\pi\sqrt{\tau_r\tau_m}$ . Introducing the frequencies  $\omega^{(0)} \equiv 2\pi/\mathcal{T}_m^{(0)}$  and  $\omega_m \equiv 1/\tau_m$  corresponding to the two time scales  $\mathcal{T}_m^{(0)}$  and  $\tau_m$ , respectively, the frequency  $\omega \equiv 2\pi/\mathcal{T}_m$  is represented as  $\omega = \sqrt{[\omega^{(0)}]^2 - \omega_m^2/4}$  corresponding to the time-oscillation period (58). In this expression for the frequency  $\omega$  the time oscillations occur only when the condition  $[\omega^{(0)}]^2 > \omega_m^2/4$  is satisfied. The existence of these two time scale  $\tau_m$  and  $\tau_r$  is therefore essential for the time-oscillatory behavior with the frequency  $\omega$ , noting that there is no time-oscillation in the over-damped case containing only  $\tau_r$ .

In Sects. 5.2 and 5.3, we investigate properties of  $G(t)$  in more detail, including its time-oscillating behavior, for (A) the work done in the laboratory frame ( $\vartheta = 1$ ), and (B) the work done in the comoving frame ( $\vartheta = 0$ ), separately. We will also compare those results with those for the over-damped case. For this purpose, we now calculate  $G(t)$  explicitly in the over-damped case. In our previous paper [7], we already calculated the work distribution function  $P_w^{(0)}(W, t)$  for the over-damped case, which is given by

$$P_w^{(0)}(W, t) = \frac{1}{\sqrt{4\pi\alpha\beta v^2[t - t_0 - \tau_r(1 - b_i)]}}$$

$$\times \exp \left\{ - \frac{[W - \alpha\beta v^2(t - t_0)]^2}{4\alpha\beta v^2[t - t_0 - \tau_r(1 - b_t)]} \right\} \tag{59}$$

with  $b_t \equiv \exp[-(t - t_0)/\tau_r]$  in the case of a nonequilibrium steady state initial distribution function  $f^{(0)}(x_i, t_0) = \sqrt{\beta\kappa/(2\pi)} \exp[-\beta\kappa(x_i - vt_0 + v\tau_r)^2/2]$  for the particle position  $x_i$  for the over-damped case at the initial time  $t_0$ .<sup>9</sup> Using (59), and defining, [cf. (49)],  $G^{(0)}(t) \equiv (\partial/\partial W) \ln[P_w^{(0)}(W, t)/P_w^{(0)}(-W, t)]$ , we have

$$G^{(0)}(t) = 1 + \frac{\tau_r(1 - b_t)}{t - t_0 - \tau_r(1 - b_t)}, \tag{60}$$

which gives  $G(t)$  for the over-damped case [24]. Note that (60) implies  $G^{(0)}(t) \xrightarrow{t \rightarrow +\infty} 1 + \tau_r/(t - t_0 - \tau_r)$ , which is consistent with (56), since  $\tau_m$  is zero for the over-damped case.

### 5.2 $G(t)$ in the Laboratory Frame

In this subsection we consider  $G(t)$  given by (55)—which depends on the parameter  $\vartheta$  to specify a frame via  $\Omega_t$ —for the work done in the laboratory frame, i.e. for  $\vartheta = 1$ . In Sect. 5.2, as well as in Sect. 5.3, we use the parameter values  $\alpha = \kappa = 1$  and set the initial time  $t_0 = 0$ , i.e.  $\tau_r = 1$  as a time unit and  $m/m^* = 4\tau_m$  as the scaled mass.

Figure 2 shows  $G(t)$  given by (55) as a function of time  $t$  for the scaled masses  $m/m^* = 0$  (over-damped case), 0.999, 2, 4, 8, 20 and 40. The graphs of  $G(t)$  all converge to 1 in the long time limit  $t \rightarrow +\infty$ , as required by the asymptotic fluctuation theorem (48), i.e. by (50).

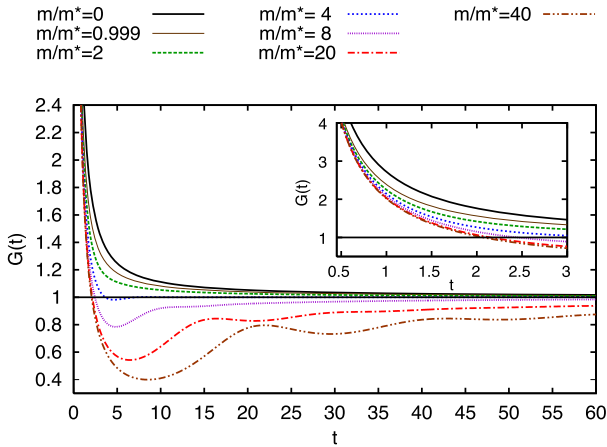
We now discuss in some detail the properties of Fig. 2. This figure shows that  $G(t)$  for nonzero masses is always smaller than in the over-damped case of zero mass. In the over-damped case,  $G(t)$  decreases monotonically to the final value 1 from  $+\infty$  at the initial time. A similar behavior is still observed for small masses (e.g. see the graph for  $m/m^* = 0.999$  in Fig. 2). It may also be noted that for small nonzero masses the relaxation of  $G(t)$  to its final value 1 is faster than in the over-damped case (e.g. see the graphs for  $m/m^* = 2$  and 4 in Fig. 2). This feature can be explained by the second term on the right-hand side of (56), since the absolute value  $|\tau_r - \tau_m|$  of the numerator of this term is smaller for  $\vartheta = 1$  than the corresponding over-damped value  $\tau_r$  in the case of  $0 < m/m^* < 8$ , using that  $|\tau_r - \tau_m| < \tau_r$ . Moreover, Fig. 2 shows that for large masses (e.g. see the graphs for  $m/m^* > 4$  in Fig. 2),  $G(t)$  is smaller than 1 for long times, while  $G(t)$  is always larger than 1 in the over-damped case. This is because the second term on the right-hand side of (56) is negative for  $\tau_r < \tau_m$  (i.e.  $m/m^* > 4$ ), when  $\vartheta = 1$  and  $t > t_0 + \tau_r - \tau_m$ .

A time-oscillatory behavior of  $G(t)$  is clearly visible in Fig. 2 for large masses, i.e. for  $m \gg m^*$ . To show more clearly the time-oscillatory behavior of  $G(t)$  for  $m > m^*$  as opposed to for  $m < m^*$ , we plotted in Fig. 3 the absolute value of the deviation<sup>10</sup>

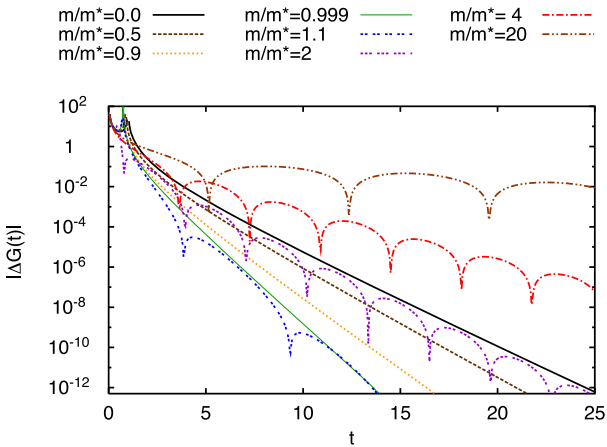
$$\Delta G(t) \equiv G(t) - 1 - \frac{\tau_r - \tau_m \vartheta^2}{t - t_0 - \tau_r + \tau_m \vartheta^2} \tag{61}$$

<sup>9</sup>Note that the work distribution function (59) approaches the distribution function (47) in the long time limit  $t \rightarrow +\infty$  because of  $t - t_0 - \tau_r(1 - b_t) \xrightarrow{t \rightarrow +\infty} t - t_0$ .

<sup>10</sup>We note that in this subsection we use the function (61) for  $\vartheta = 1$ , while in Sect. 5.3 we use the function (61) for  $\vartheta = 0$ .

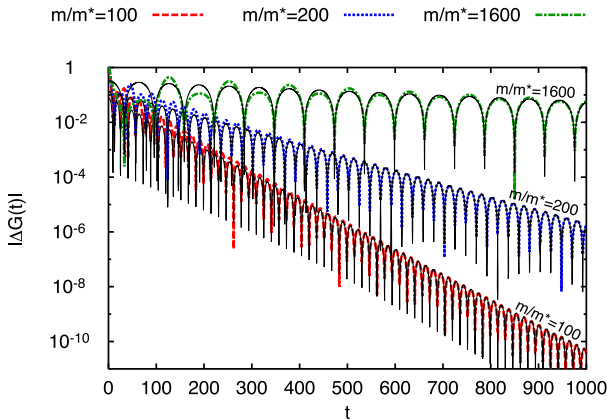


**Fig. 2** (Color online) Graphs of  $G(t) = (\partial/\partial W) \ln[P_w(W, t)/P_w(-W, t)]$  as a function of time  $t$  for the work done in the laboratory frame ( $\vartheta = 1$ ) in the case of a nonequilibrium steady state initial condition for  $t \in [0, 60]$ . *Inset*: graphs of  $G(t)$  in a short time period for  $t \in [0, 3]$ . *Lines* in these graphs correspond to parameter values of the scaled masses  $m/m^* = 0$  (over-damped case), 0.999, 2, 4, 8, 20 and 40 as indicated above this figure, and we used parameter values  $\alpha = \kappa = 1$  (so that  $\tau_r = 1$  for a unit time and also  $m/m^* = 4\tau_m$  as the scaled mass, with  $m^* = 1/4$ ) and  $t_0 = 0$



**Fig. 3** (Color online) Linear-log plots of absolute value  $|\Delta G(t)|$  of the function  $\Delta G(t) = G(t) - 1 - (\tau_r - \tau_m)/(t - \tau_r + \tau_m)$  as a function of time  $t$  for the work done in the laboratory frame in the case of a nonequilibrium steady state initial condition. *Lines* in these graphs correspond to parameter values of the scaled masses  $m/m^* = 0, 0.5, 0.9, 0.999, 1.1, 2, 4$  and  $20$  as indicated above this figure. The minima of the oscillations of  $|\Delta G(t)|$  for  $m/m^* > 1$  are actually zero, which is not indicated in this figure and Figs. 4, 6 and 7. We use the same parameter values  $\alpha, \kappa$  and  $t_0$  as in Fig. 2

of  $G(t)$  from its asymptotic form (56) as a function of time  $t \in [0, 25]$  for the cases of  $m/m^* = 0, 0.5, 0.9, 0.999, 1.1, 2, 4$  and  $20$ . To illustrate the long time behavior of  $|\Delta G(t)|$  in more detail, we also show in Fig. 4 the absolute value  $|\Delta G(t)|$  of  $\Delta G(t)$  as functions of  $t \in [0, 1000]$  for the scaled masses  $m/m^* = 100, 200$  and  $1600$  as linear-log plots. The deviation  $\Delta G(t)$  goes to zero when  $t \rightarrow +\infty$  because of the asymptotic fluctuation theorem



**Fig. 4** (Color online) Long time behavior of  $|\Delta G(t)|$  as a function of time  $t$  as linear–log plots for the work done in the laboratory frame in the case of a nonequilibrium steady state initial condition. Here, we use the same parameter values  $\alpha$ ,  $\kappa$  and  $t_0$  as in Fig. 2. Broken, dotted and dash-dotted lines in these graphs correspond to parameter values of the scaled masses  $m/m^* = 100, 200$  and  $1600$ , respectively. Solid lines are fits of  $|\Delta G(t)|$  to the function (62) using Table 1, together with the time-oscillation period (58), but they are visually indistinguishable from the graphs of  $|\Delta G(t)|$  except for short times

(50). In Figs. 3 and 4, it is important to note that there is no time-oscillation of  $\Delta G(t)$  for  $0 \leq m/m^* < 1$ , while we do observe time-oscillations of  $\Delta G(t)$  for  $m/m^* > 1$ , in agreement with a critical mass (57), above which  $G(t)$  oscillates in time. The decay of  $|\Delta G(t)|$  to zero as a function of  $t$  is faster for larger masses for  $0 \leq m/m^* < 1$  (cf. Fig. 3), but slower for larger masses for  $m/m^* > 1$  (cf. Figs. 3 and 4).

To check that the time oscillation period  $\mathcal{T}_m$  given by (58) indeed appears in  $G(t)$ , we fitted the data for  $\Delta G(t)$  to the function

$$\Delta G(t) \stackrel{t \rightarrow +\infty}{\sim} a e^{-bt} \sin\left(\frac{2\pi}{\mathcal{T}_m} t + c\right) \tag{62}$$

with fitting parameters  $a$ ,  $b$  and  $c$  in Fig. 4. The values of the fitting parameters  $a$ ,  $b$  and  $c$  are given in Table 1. The function (62) is then sufficiently close to  $\Delta G(t)$  over many time-oscillation periods (except for short times), to suggest that the time-oscillations of  $G(t)$  may well have the same origin as those in the position  $\tilde{x}_s^*$ . Similarly for Fig. 3, using the fitting function (62) we can also check that the time-oscillation periods of  $|\Delta G(t)|$  in this figure are given by (58). We fully realize that Figs. 3 and 4 are not enough to specify convincingly the form of the decay of  $\Delta G(t)$ . In (62) we assumed an exponential decay by a factor  $a \exp(-bt)$ , which seems to fit reasonably well the data in Fig. 4. However, values of the fitting parameters  $a$  and  $b$  shown in Table 1 appear to vary non-negligibly if we fit data including longer time periods than the ones shown in Fig. 4. In this sense, at this stage, the exponential factor in (62) should be regarded only as a convenience to check numerically the time oscillation period  $\mathcal{T}_m$  appearing in  $\Delta G(t)$ , rather than claiming an asymptotic exponential decay of  $\Delta G(t)$  of the form (62).

### 5.3 $G(t)$ in the Comoving Frame

Here we consider  $G(t)$  for the work done in the comoving frame, namely the case of  $\vartheta = 0$ , in which the work includes effects of a d’Alembert-like force.



**Table 1** Values of the fitting parameters  $a$ ,  $b$  and  $c$  for the function (62) plotted in Figs. 4 and 7 for the parameter values  $\alpha = \kappa = 1$

Frame ( $\vartheta$ )	$m/m^*$	$\mathcal{T}_m$	$a$	$b$	$c$
Laboratory (1)	100	31.6	-0.077	0.021	4.4
Laboratory (1)	200	44.5	-0.14	0.011	4.5
Laboratory (1)	1600	125.7	-0.32	0.0017	4.6
Comoving (0)	100	31.6	0.0038	0.021	4.6
Comoving (0)	200	44.5	0.0035	0.011	4.6
Comoving (0)	1600	125.7	0.0032	0.0024	4.7

**Fig. 5** (Color online) Graphs of  $G(t) = (\partial/\partial W) \ln[P_w(W, t)/P_w(-W, t)]$  as a function of time  $t$  for the work done in the comoving frame ( $\vartheta = 0$ ) in the case of a nonequilibrium steady state initial condition for  $t \in [0, 60]$ . *Inset*: graphs of  $G(t)$  in a short time period for  $t \in [0, 6]$ . *Lines* in these graphs correspond to parameter values of the scaled masses  $m/m^* = 0$  (over-damped case), 0.999, 2, 4, 8, 20 and 40 as indicated above this figure and we use the same parameter values  $\alpha, \kappa$  and  $t_0$  as in Fig. 2

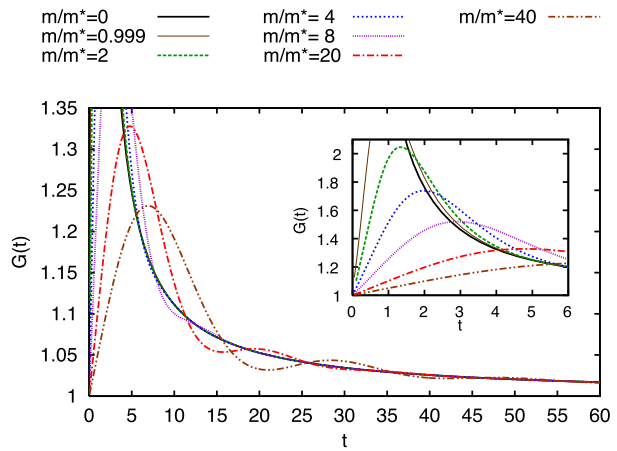
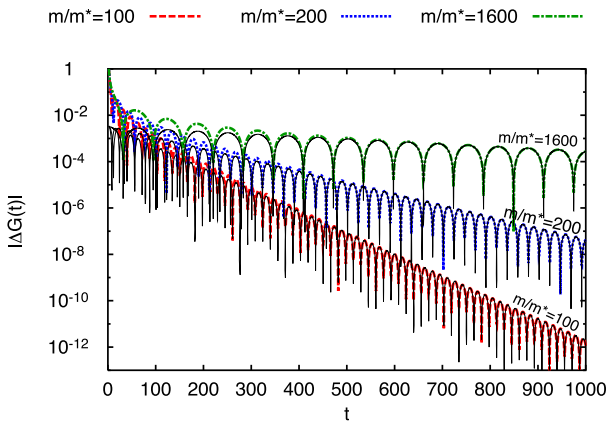
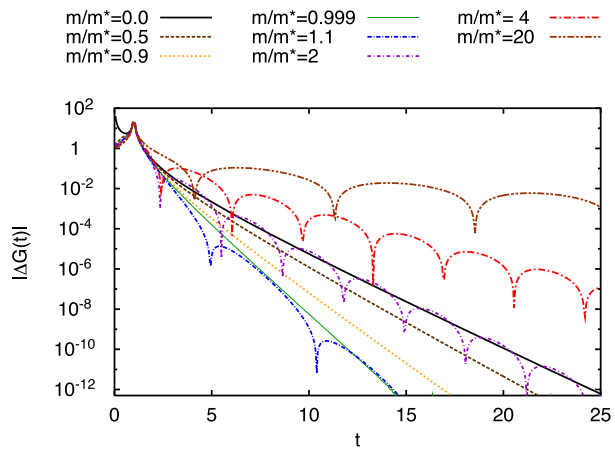


Figure 5 shows graphs of  $G(t)$  given by (55) as a function of time  $t$ . We chose the same masses as in Fig. 2, namely  $m/m^* = 0$  (over-damped case), 0.999, 2, 4, 8, 20 and 40 with the critical mass  $m^* = 1/4$ . It is clear that in Fig. 5 graphs of  $G(t)$  approach 1 as  $t \rightarrow +\infty$ , confirming the asymptotic fluctuation theorem (48).

Comparing Fig. 2 with Fig. 5, a dramatic difference in the behavior of  $G(t)$  in the two frames is clearly visible. First, a striking frame-dependence of  $G(t)$  is that for any nonzero mass,  $G(t)$  in the comoving frame starts from a finite value at the initial time  $t_0 (=0)$  and is always larger than 1, in fact going through a maximum to its final value 1. This contrary to in the laboratory frame where  $G(t)$  diverges for  $t \rightarrow t_0 + 0$  and can be smaller than 1 for large masses and long times as discussed in Sect. 5.2. Another remarkable point is that, different from in the laboratory frame, shown in Fig. 2,  $G(t)$  converges to the over-damped line, much before converging to its final value 1, as shown in Fig. 5. This feature can be explained by the asymptotic form (56) of  $G(t)$ , whose right-hand side is independent of the mass  $m$  in the comoving frame ( $\vartheta = 0$ ), so a relaxation behavior of  $G(t)$  to its final value 1 in this frame should be close to that of the over-damped case.

Now, we discuss the time-oscillatory behavior of  $G(t)$  in the comoving frame. We note that in the comoving frame the approach of  $G(t)$  to its final value 1 is via oscillations around the over-damped line, contrary to in the laboratory frame where this approach is unrelated to the over-damped line. Such time-oscillations are already visible for large masses  $m \gg m^*$  in Fig. 5, but to show them in a more magnified way, we plotted in Fig. 6 the absolute value

**Fig. 6** (Color online) Linear–log plots of the absolute value  $|\Delta G(t)|$  of the function  $\Delta G(t) = G(t) - 1 - \tau_r/(t - \tau_r)$  as a function of time  $t$  for the work done in the comoving frame in the case of a nonequilibrium steady state initial condition. Lines in these graphs correspond to parameter values of the scaled masses  $m/m^* = 0, 0.5, 0.9, 0.999, 1.1, 2, 4$  and  $20$  as indicated above this figure and we use the same parameter values  $\alpha, \kappa$  and  $t_0$  as in Fig. 2



**Fig. 7** (Color online) Long time behavior of  $|\Delta G(t)|$  as a function of time  $t$  as linear–log plots for the work done in the comoving frame in the case of a nonequilibrium steady state initial condition. Here, we use the same parameter values  $\alpha, \kappa$  and  $t_0$  as in Fig. 2. Broken, dotted and dash-dotted lines in these graphs correspond to parameter values of the scaled masses  $m/m^* = 100, 200$  and  $1600$ , respectively. Solid lines are fits of  $|\Delta G(t)|$  to the function (62) using Table 1, and they are visually indistinguishable from the graphs of  $|\Delta G(t)|$  except for short times

$|\Delta G(t)|$  of the function  $\Delta G(t)$  defined by (61) as a function of time  $t$  as linear–log plots. Here, we plotted data for the scaled masses  $m/m^* = 0, 0.5, 0.9, 0.999, 1.1, 2, 4$  and  $20$  and for the time period  $t \in [0, 25]$ . It is shown in Fig. 6 that time-oscillations of  $\Delta G(t)$  occur for  $m/m^* > 1$  but not for  $0 \leq m/m^* < 1$ . Moreover,  $\Delta G(t)$  for  $0 \leq m/m^* < 1$  decays faster, while  $\Delta G(t)$  for  $m/m^* > 1$  decays slower with time, for increasing mass. These features are similar to those in the laboratory frame.

In Fig. 7 we show linear–log plots of  $|\Delta G(t)|$  as functions of  $t$  for longer times  $t \in [0, 1000]$  and for larger masses  $m/m^* = 100, 200$  and  $1600$  than in Fig. 6. Comparing this figure in the comoving frame with the corresponding Fig. 4 in the laboratory frame, we can see that the time-oscillation amplitudes of the function  $\Delta G(t)$  in the comoving frame are much smaller than the corresponding ones in the laboratory frame, except for short times. This should be noted as an important frame-dependence in the behavior of  $G(t)$ .

The time-oscillation periods appearing in Figs. 6 and 7 can be checked by fitting the data again to the function (62) with the time-oscillation period (58). We only show such fitting lines for Fig. 7 using the fitting parameters  $a$ ,  $b$  and  $c$  of Table 1. Like for the fitting lines in Fig. 4, the parameter values of  $a$  and  $b$  in Table 1 in the comoving frame also appear to vary non-negligibly for data for a longer time period than that shown in Fig. 7. Therefore, as for Fig. 4, the fitting lines in Fig. 7 should not be regarded as evidence for the exponential decay in the fitting function (62). However, the fits of their time-oscillation periods of  $|\Delta G(t)|$  to the function (62) in Fig. 7 are satisfactory, which suggests again that the time-oscillations of  $G(t)$  have the same origin as those in the position  $\tilde{x}_s^*$ , like in the laboratory frame.

### 6 Summary and Remarks

As a summary of this paper, we have discussed inertial effects related to the particle mass  $m$  in nonequilibrium work distribution functions and their associated fluctuation theorems for a dragged Brownian particle model confined by a harmonic potential using a path integral approach for all times: asymptotic as well as finite. We considered two kinds of work: the work  $\mathcal{W}_l$  done in the laboratory frame and the work  $\mathcal{W}_c$  done in the comoving frame and we calculated the distribution functions  $P_w(W, t)$  for them. Using the distributions for the work in the different frames we analytically proved, for any initial condition, an asymptotic work fluctuation theorem, which has the same form in both the frames. This contrasts with what happens for finite times, when for a nonequilibrium steady state initial condition there are major differences between the work fluctuations in the laboratory and comoving frames. This was discussed, using the quantity  $G(t) \equiv (\partial/\partial W) \ln[P_w(W, t)/P_w(-W, t)]$ , which approaches the value 1 in the long time limit  $t \rightarrow +\infty$  by the asymptotic fluctuation theorem. The  $G(t)$  for the work  $\mathcal{W}_c$  done in the comoving frame is larger than 1 at all times and converges to the corresponding over-damped value much before converging to its final value 1. On the other hand, the  $G(t)$  for the work  $\mathcal{W}_l$  done in the laboratory frame can be smaller than 1 for sufficiently large times and masses, and the relaxation behavior of  $G(t)$  to its final value 1 is very different from that for the over-damped case, even for long times. As one of the significant effects for finite times, we also discussed the existence of a critical mass  $m^*$ , so that for the mass  $m > m^*$  a time-oscillatory behavior appears in  $G(t)$  in both frames.

In the remainder of this section, we make some remarks on the contents in the main text of this paper.

(1) We have discussed in this paper differences between the works  $\mathcal{W}_l$  and  $\mathcal{W}_c$ , which originate in a frame dependence of the kinetic energy difference due to the d’Alembert-like force as we discussed in Sect. 3.1. In contrast to the work and the kinetic energy difference, the heat (as well as the potential energy difference) has the same value in the laboratory and the comoving frame even in the inertial case. Note that the two works  $\mathcal{W}_l$  and  $\mathcal{W}_c$  have the same average value in the nonequilibrium steady state, because their difference can be represented as a “boundary term”

$$\mathcal{W}_l - \mathcal{W}_c = m(\dot{x}_t - \dot{x}_{t_0})v \tag{63}$$

depending on a difference between the two boundary values of  $\dot{x}_s$  at the final time  $s = t$  and the initial time  $s = t_0$  only, so that the average of this boundary term  $m(\dot{x}_t - \dot{x}_{t_0})v$  is zero in the nonequilibrium steady state. Nevertheless, this difference  $m(\dot{x}_t - \dot{x}_{t_0})v$  between  $\mathcal{W}_l$  and  $\mathcal{W}_c$  causes dramatic differences in the work fluctuations at finite times, as shown in Sects. 5.2 and 5.3 of this paper.

**Table 2** Correspondences between the dragged Brownian particle model described by (3) and the torsion pendulum model described by (64)

Brownian particle	$x_s$	$m$	$\alpha$	$\kappa$	$\kappa v$
Torsion pendulum	$\theta_s$	$I$	$\nu$	$C$	$\mu$

(2) In a different nonequilibrium model described by a linear Langevin equation, [18, 19] considered the motion of a torsion pendulum under an external torque in a fluid. The corresponding Langevin equation for the angular displacement  $\theta_s$  of the pendulum at time  $s$  in this system is then given by

$$I \frac{d^2\theta_s}{ds^2} = -\nu \frac{d\theta_s}{ds} - C\theta_s + M_s + \zeta_s \tag{64}$$

where  $I$  is the total moment of inertia of the mass,  $\nu$  is the viscous damping,  $C$  the elastic torsional stiffness of the pendulum,  $M_s$  the external torque, and  $\zeta_s$  the Gaussian-white random force. For this model, [18, 19] considered the case of a linear torque of

$$M_s = \mu s \tag{65}$$

with a force constant  $\mu$ . It is important to note that (64) with the force (65) has mathematically the same form as the Langevin equation (3) with the correspondences shown in Table 2. Based on these correspondences between the two models, for example, there should be a critical value  $I^*$  of the total moment of inertia above which a similar time-oscillatory behavior occurs in the pendulum model, like above the critical mass  $m^*$  in the dragged Brownian particle model treated in this paper.

For the pendulum system, [18, 19] considered the work  $\mathcal{W}_p$  done by the external torque  $M_s$  on the pendulum ( $p$ ). This work is given there by

$$\mathcal{W}_p = \int_{t_0}^t ds (M_s - M_{t_0}) \frac{d\theta_s}{ds}. \tag{66}$$

Using (65) and the correspondences in Table 2, this work corresponds to a quantity for our dragged Brownian particle model, viz.

$$\mathcal{W}_p \longleftrightarrow \int_{t_0}^t ds \kappa v (s - t_0) \frac{dx_s}{ds} = \mathcal{W}_l + \kappa v (t - t_0) \left[ x_t - \frac{1}{2} v (t + t_0) \right] \tag{67}$$

which is clearly different from the works  $\mathcal{W}_l$  and  $\mathcal{W}_c$  discussed in this paper. In other words,  $\mathcal{W}_l$ ,  $\mathcal{W}_c$  and  $\mathcal{W}_p$  give physically different kinds of work in nonequilibrium steady states described by a mathematically identical Langevin equation in a dynamical sense. We note that our  $\mathcal{W}_l$  and  $\mathcal{W}_c$  are consequences of the generalized Onsager–Machlup theory in [7]. We reserve, for a future publication, a general discussion on fluctuation theorems for different models.

(3) As another nonequilibrium model described by a linear Langevin equation, [25] considered electric circuit models. In that case the system is described by a first-order linear Langevin equation, which has the same form as the over-damped Langevin equation for the dragged Brownian particle model. As a generalization of these electric circuit models, an inertial effect in the electric circuit can be introduced by including its self-induction.

A generalization of the arguments of [25] to the case including the self-induction, as well as a discussion of the effects of self-induction on the nonequilibrium work (and heat) fluctuations, will be addressed in a future paper. Especially, it would be interesting to observe whether there is a critical value of the self-induction, above which similar oscillatory effects occur, as appear above the critical mass in the inertial case in this paper.

(4) The critical mass  $m^*$  discussed in this paper for work fluctuations also appears in the dynamics of the average position  $\langle x_s \rangle$ . In order to discuss this point, we note that taking the average of (3), the average position  $\langle x_s \rangle$  of the particle at time  $s$  satisfies

$$m \frac{d^2 \langle x_s \rangle}{ds^2} = -\alpha \frac{d \langle x_s \rangle}{ds} - \kappa (\langle x_s \rangle - vs) \tag{68}$$

using  $\langle \zeta_s \rangle = 0$ . Using  $v_{\pm}$  defined by (20), the solution of (68) is given by

$$\langle x_s \rangle = v(s - \tau_r) + C' e^{-v_+ s} + C'' e^{-v_- s} \tag{69}$$

where the constants  $C'$  and  $C''$  are determined by the average initial conditions  $\langle x_{t_0} \rangle$  and  $\langle \dot{x}_{t_0} \rangle$  and are given by

$$C' = -\frac{v_- e^{v_+ t_0}}{v_+ - v_-} [\langle x_{t_0} \rangle - v(t_0 - \tau_r)] - \frac{e^{v_+ t_0}}{v_+ - v_-} (\langle \dot{x}_{t_0} \rangle - v), \tag{70}$$

$$C'' = \frac{v_+ e^{v_- t_0}}{v_+ - v_-} [\langle x_{t_0} \rangle - v(t_0 - \tau_r)] + \frac{e^{v_- t_0}}{v_+ - v_-} (\langle \dot{x}_{t_0} \rangle - v). \tag{71}$$

Since the  $v_{\pm}$  include nonzero imaginary parts for  $m > m^*$ , a time-oscillatory behavior appears in the average position  $\langle x_s \rangle$  for masses above this critical mass  $m^*$ . This kind of phenomenon was discussed for a damped oscillator model [21], but its effect on fluctuations in a nonequilibrium steady state has not been discussed to the best of our knowledge.

In [7], we discussed that in the over-damped case, the most probable path, which is a solution of the Euler–Lagrange equation for the Lagrangian function in the Onsager–Machlup theory, is expressed as a combination of forward and backward paths. This is also true in the inertial case, in which the most probable path is given by a solution of the “Euler–Lagrange” equation (16) for  $\lambda = 0$ . To show this, we note that the exponentially decaying terms  $\exp(-v_+ s)$  and  $\exp(-v_- s)$  on the right-hand side of (69) refer to a forward path. We can also introduce the corresponding backward path, as a combination of exponentially divergent terms  $\exp(v_+ s)$  and  $\exp(v_- s)$ . A combination of these forward and backward paths gives then the most probable path  $\{x_s^*\}_{s \in [t_0, t]}$  for  $\lambda = 0$ , i.e. (18).

(5) There is still the open question of an analytical discussion of the asymptotic form of  $\Delta G(t)$  with the time-oscillations shown in Figs. 3, 4, 6 and 7. In this paper we only analyzed  $\Delta G(t)$  numerically by fitting it to the function (62), but in principle, such analytical information on  $\Delta G(t)$  is contained in the general form (55) of  $G(t)$ .

(6) We have considered the asymptotic fluctuation theorem for work in this paper. We now address very briefly its connection with other fluctuation theorems.

(6a) One of the other fluctuation theorems is the transient fluctuation theorem [2]. This fluctuation theorem was already derived and discussed for a dragged Brownian particle model with inertia in [7]. There, we derived transient fluctuation theorems, not only for the same works as those in this paper, but also for an energy loss by friction. Different from the work, the distribution function for the energy loss by friction does *not* satisfy an asymptotic fluctuation theorem.

(6b) Another important fluctuation theorem is the extended heat fluctuation theorem [22, 23, 26]. In [7] we gave a simple derivation of this fluctuation theorem, based on the assumptions that (A) a correlation between the work and the energy difference at time  $t$  (as well as a correlation between the energies at the initial time  $t_0$  and the final time  $t$ ) disappears in the long time limit  $t \rightarrow +\infty$ , (B) the work distribution function approaches a Gaussian distribution asymptotically in time and satisfies the asymptotic fluctuation theorem, and (C) the distribution function  $P_e(E)$  for energy  $E$  is canonical-like, namely  $P_e(E) \approx \exp(-\beta E)$  for  $E > 0$ . The same derivation could be applied to all models which satisfy these three conditions (A), (B) and (C). In particular, using this derivation, one can derive an analytical expression for the asymptotic heat distribution function itself, as well as the extended heat fluctuation theorem not only for the over-damped case, as was done in [7], but also for the inertial case.

**Acknowledgements** We gratefully acknowledge financial support of the National Science Foundation, under award PHY-0501315.

**Appendix 1: Asymptotic Property for the Matrix  $\Lambda_t$**

In this appendix we prove (46) for the matrix  $\Lambda_t$ . To show this equation in a simple way, without losing generality we take the origin of time at  $(t_0 + t)/2$  so that the initial time is given by  $t_0 = -t$ , only in this appendix.

To consider the structure of the matrix  $\Lambda_t$  defined by (39) in the long time limit  $t \rightarrow +\infty$ , we first calculate the asymptotic form of the matrix  $\Gamma \Phi_t \Gamma$ , which is an essential element of the matrix  $\Lambda_t$ . For this purpose we note

$$\mathbf{K}_s \mathbf{K}_s^T = \begin{pmatrix} e^{2v_+ s} & e^{(v_+ + v_-)s} & e^{(v_+ - v_-)s} & 1 \\ e^{(v_+ + v_-)s} & e^{2v_- s} & 1 & e^{-(v_+ - v_-)s} \\ e^{(v_+ - v_-)s} & 1 & e^{-2v_- s} & e^{-(v_+ + v_-)s} \\ 1 & e^{-(v_+ - v_-)s} & e^{-(v_+ + v_-)s} & e^{-2v_+ s} \end{pmatrix}. \tag{72}$$

Inserting (72) into (41) and using the relation  $t_0 = -t$  we obtain

$$\Phi_t = 2 \begin{pmatrix} \frac{\sinh(2v_+ t)}{2v_+} & \frac{\sinh[(v_+ + v_-)t]}{v_+ + v_-} & \frac{\sinh[(v_+ - v_-)t]}{v_+ - v_-} & t \\ \frac{\sinh[(v_+ + v_-)t]}{\sinh[(v_+ - v_-)t]} & \frac{\sinh(2v_- t)}{2v_-} & t & \frac{\sinh[(v_+ - v_-)t]}{\sinh[(v_+ + v_-)t]} \\ \frac{v_+ + v_-}{\sinh[(v_+ - v_-)t]} & t & \frac{\sinh(2v_- t)}{2v_-} & \frac{\sinh[(v_+ - v_-)t]}{\sinh[(v_+ + v_-)t]} \\ t & \frac{\sinh[(v_+ - v_-)t]}{v_+ - v_-} & \frac{\sinh[(v_+ + v_-)t]}{v_+ + v_-} & \frac{\sinh(2v_+ t)}{2v_+} \end{pmatrix} \tag{73}$$

in terms of the hyperbolic function  $\sinh(x)$ . Equations (34) and (73) lead to

$$\Gamma \Phi_t \Gamma = \begin{pmatrix} \Psi_t & 0_2 \\ 0_2 & 0_2 \end{pmatrix} \tag{74}$$

where  $0_2$  is the  $2 \times 2$  null matrix, and the  $2 \times 2$  matrix  $\Psi_t$  is given by

$$\Psi_t \equiv 4 \begin{pmatrix} v_+ \sinh(2v_+ t) & \frac{2v_+ v_-}{v_+ + v_-} \sinh[(v_+ + v_-)t] \\ \frac{2v_+ v_-}{v_+ + v_-} \sinh[(v_+ + v_-)t] & v_- \sinh(2v_- t) \end{pmatrix}$$

$$\begin{aligned}
 & \stackrel{t \rightarrow +\infty}{\sim} 2 \begin{pmatrix} \nu_+ e^{2\nu_+ t} & \frac{2\nu_+ \nu_-}{\nu_+ + \nu_-} e^{(\nu_+ + \nu_-)t} \\ \frac{2\nu_+ \nu_-}{\nu_+ + \nu_-} e^{(\nu_+ + \nu_-)t} & \nu_- e^{2\nu_- t} \end{pmatrix} \\
 & = 2 \begin{pmatrix} e^{\nu_+ t} & 0 \\ 0 & e^{\nu_- t} \end{pmatrix} \begin{pmatrix} \nu_+ & \frac{2\nu_+ \nu_-}{\nu_+ + \nu_-} \\ \frac{2\nu_+ \nu_-}{\nu_+ + \nu_-} & \nu_- \end{pmatrix} \begin{pmatrix} e^{\nu_+ t} & 0 \\ 0 & e^{\nu_- t} \end{pmatrix}. \tag{75}
 \end{aligned}$$

Here, we used the positivity  $\text{Re}\{\nu_{\pm}\} > 0$  of the real part of  $\nu_{\pm}$  (assuming a nonzero mass  $m \neq 0$  and a nonzero spring constant  $\kappa \neq 0$ ) and also  $\sinh(at) \stackrel{t \rightarrow +\infty}{\sim} (1/2)\exp(at)$  for any number  $a$  with the positive real part  $\text{Re}\{a\} > 0$ .

Second, we obtain a simplified form of the matrix  $A_t^{-1}$  in the long time limit, which is another essential element of the matrix  $\Lambda_t$ . Noting again that the real part of the number  $\nu_{\pm}$  is (strictly non-zero) positive and the initial time is given by  $t_0 = -t$ , we obtain the asymptotic form of the matrix  $A_t$  defined by (25) as

$$A_t \stackrel{t \rightarrow +\infty}{\sim} \begin{pmatrix} 0_2 & A_t^{(1)} \\ A_t^{(2)} & 0_2 \end{pmatrix} \tag{76}$$

for the long time limit  $t \rightarrow +\infty$ . Here,  $0_2$  is the  $2 \times 2$  null matrix, and  $A_t^{(j)}$ ,  $j = 1, 2$  are defined by

$$A_t^{(1)} \equiv \begin{pmatrix} e^{\nu_- t} & e^{\nu_+ t} \\ -\nu_- e^{\nu_- t} & -\nu_+ e^{\nu_+ t} \end{pmatrix}, \tag{77}$$

$$A_t^{(2)} \equiv \begin{pmatrix} e^{\nu_+ t} & e^{\nu_- t} \\ \nu_+ e^{\nu_+ t} & \nu_- e^{\nu_- t} \end{pmatrix}. \tag{78}$$

From (76) we derive

$$A_t^{-1} \stackrel{t \rightarrow +\infty}{\sim} \begin{pmatrix} 0_2 & A_t^{(2)-1} \\ A_t^{(1)-1} & 0_2 \end{pmatrix} \tag{79}$$

where  $A_t^{(1)-1}$  and  $A_t^{(2)-1}$  are given by

$$A_t^{(1)-1} = \frac{1}{\nu_- - \nu_+} \begin{pmatrix} -\nu_+ e^{-\nu_- t} & -e^{-\nu_- t} \\ \nu_- e^{-\nu_+ t} & e^{-\nu_+ t} \end{pmatrix}, \tag{80}$$

$$A_t^{(2)-1} = \frac{1}{\nu_- - \nu_+} \begin{pmatrix} \nu_- e^{-\nu_+ t} & -e^{-\nu_+ t} \\ -\nu_+ e^{-\nu_- t} & e^{-\nu_- t} \end{pmatrix}. \tag{81}$$

Equation (79) gives an asymptotic form for the matrix  $A_t^{-1}$ .

Finally, using (39), (74) and (79) we obtain the asymptotic form of the matrix  $\Lambda_t$  as

$$\Lambda_t \stackrel{t \rightarrow +\infty}{\sim} \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & [A_t^{(2)-1}]^T \Psi_t A_t^{(2)-1} \end{pmatrix}. \tag{82}$$

Here, using (75) and (81) the non-vanishing matrix elements of the matrix (82) are given by

$$[A_t^{(2)-1}]^T \Psi_t A_t^{(2)-1} \stackrel{t \rightarrow +\infty}{\sim} 2 \begin{pmatrix} \frac{\nu_+ \nu_-}{\nu_+ + \nu_-} & 0 \\ 0 & \frac{1}{\nu_+ + \nu_-} \end{pmatrix} = \frac{2}{\alpha} \begin{pmatrix} \kappa & 0 \\ 0 & m \end{pmatrix} \tag{83}$$

where we used  $v_+ + v_- = \alpha/m$  and  $v_+ v_- = \kappa/m$ . By (82) and (83) we obtain

$$\lim_{t \rightarrow +\infty} \Lambda_t = \frac{2}{\alpha} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa & 0 \\ 0 & 0 & 0 & m \end{pmatrix}. \tag{84}$$

Equation (84) shows that the matrix  $\Lambda_t$  approaches a time-independent constant matrix in the long time limit  $t \rightarrow +\infty$ . Therefore, the matrix  $\Lambda_t/(t - t_0)$  approaches the  $4 \times 4$  null matrix in the long time limit  $t \rightarrow +\infty$ , implying that the condition (46) is satisfied.

### Appendix 2: Work Distribution for the Nonequilibrium Steady State

In this appendix we give a derivation of (52) for the work distribution function  $P(W, t)$  in the case of the nonequilibrium steady state initial condition (51).

First, we note that the initial distribution function (51) can be written in the form

$$f(x_i, p_i, t_0) = \frac{\beta}{2\pi} \sqrt{\frac{\kappa}{m}} \exp \left\{ -\frac{\alpha\beta}{4} [\mathbf{B}_{if}^{(1)}]^T \Lambda^{(0)} \mathbf{B}_{if}^{(1)} \right\}, \tag{85}$$

using (26) and (54). Equation (85) means that the initial distribution function  $f(x_i, p_i, t_0)$  is Gaussian for the components of the vector  $\mathbf{B}_{if}^{(1)}$ . Using (85) and the vector  $\mathbf{B}_{if}^{(1)}$  given by (26), the work distribution function (44) can be represented by

$$\begin{aligned} P_w(W, t) &= \frac{C_\varepsilon m \beta}{4\pi v} \sqrt{\frac{\kappa m}{\pi \alpha \beta (t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})}} \\ &\times \iiint \int d\mathbf{B}_{if}^{(1)} \exp \left[ -\frac{1}{4} \alpha \beta [\mathbf{B}_{if}^{(1)}]^T (\Lambda^{(0)} + \Lambda_t) \mathbf{B}_{if}^{(1)} \right] \\ &\times \exp \left\{ -\frac{[\alpha \beta v (\boldsymbol{\eta}^T - \tau_r \mathbf{J}^T \Lambda_t) \mathbf{B}_{if}^{(1)} - W + \alpha \beta v^2 (t - t_0)]^2}{4 \alpha \beta v^2 (t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})} \right\} \end{aligned} \tag{86}$$

where we used the relation  $dx_i dp_i dx_f dp_f = m^2 d\mathbf{B}_{if}^{(1)}$  because of (26).

Now, we note that

$$\begin{aligned} &(\tilde{\mathbf{a}}^T \tilde{\mathbf{x}} + \tilde{b})^2 + \tilde{\mathbf{x}}^T \tilde{\mathcal{C}} \tilde{\mathbf{x}} \\ &= \tilde{\mathbf{x}}^T (\tilde{\mathbf{a}} \tilde{\mathbf{a}}^T + \tilde{\mathcal{C}}) \tilde{\mathbf{x}} + 2 \tilde{b} \tilde{\mathbf{a}}^T \tilde{\mathbf{x}} + \tilde{b}^2 \\ &= [\tilde{\mathbf{x}} + \tilde{b} (\tilde{\mathbf{a}} \tilde{\mathbf{a}}^T + \tilde{\mathcal{C}})^{-1} \tilde{\mathbf{a}}]^T (\tilde{\mathbf{a}} \tilde{\mathbf{a}}^T + \tilde{\mathcal{C}}) [\tilde{\mathbf{x}} + \tilde{b} (\tilde{\mathbf{a}} \tilde{\mathbf{a}}^T + \tilde{\mathcal{C}})^{-1} \tilde{\mathbf{a}}] + \tilde{b}^2 [1 - \tilde{\mathbf{a}}^T (\tilde{\mathbf{a}} \tilde{\mathbf{a}}^T + \tilde{\mathcal{C}})^{-1} \tilde{\mathbf{a}}] \end{aligned} \tag{87}$$

for any  $n$ -dimensional vectors  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{x}}$ , any scalar  $\tilde{b}$ , and any  $n \times n$  symmetric matrix  $\tilde{\mathcal{C}}$  for which the inverse matrix of  $\tilde{\mathbf{a}} \tilde{\mathbf{a}}^T + \tilde{\mathcal{C}}$  exists. Applying (87) to (86) for the case of  $\tilde{\mathbf{x}} = \mathbf{B}_{if}^{(1)}$ ,  $\tilde{\mathbf{a}} = \alpha \beta v (\boldsymbol{\eta} - \tau_r \Lambda_t \mathbf{J})$ ,  $\tilde{b} = \alpha \beta v^2 (t - t_0) - W$  and  $\tilde{\mathcal{C}} = (\alpha \beta v)^2 (t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J}) (\Lambda^{(0)} + \Lambda_t)$ , and carrying out the integral over  $\mathbf{B}_{if}^{(1)}$  ( $= \tilde{\mathbf{x}}$ ) in (86), we obtain

$$P_w(W, t) = \frac{C_\varepsilon m \beta}{4\pi v} \sqrt{\frac{\kappa m}{\pi \alpha \beta (t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})}}$$



$$\begin{aligned}
 & \times \iiint d\tilde{\mathbf{x}} \exp \left\{ -\frac{(\tilde{\mathbf{a}}^T \tilde{\mathbf{x}} + \tilde{b})^2 + \tilde{\mathbf{x}}^T \tilde{C} \tilde{\mathbf{x}}}{4\alpha\beta v^2(t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})} \right\} \\
 & = C'_w \exp \left\{ -\frac{[1 - \tilde{\mathbf{a}}^T (\tilde{\mathbf{a}}\tilde{\mathbf{a}}^T + \tilde{C})^{-1} \tilde{\mathbf{a}}]}{4\alpha\beta v^2(t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})} \tilde{b}^2 \right\} \\
 & = C'_w \exp \left\{ -\frac{1 - \Omega_t}{4\alpha\beta v^2(t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})} [W - \alpha\beta v^2(t - t_0)]^2 \right\} \tag{88}
 \end{aligned}$$

using (53) for  $\Omega_t$ . The constant  $C'_w$  in (88) can be determined by the normalization condition  $\int dW P(W, t) = 1$  and then (88) gives (52).

**Appendix 3: Asymptotic Form of  $G(t)$**

In this appendix we derive (56) for  $G(t)$ .

The essential point to derive (56) for  $G(t)$  is the asymptotic form (84), or equivalently

$$\lim_{t \rightarrow +\infty} \Lambda_t = 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\tau_r & 0 \\ 0 & 0 & 0 & \tau_m \end{pmatrix} \tag{89}$$

in terms of  $\tau_r = \alpha/\kappa$  and  $\tau_m = m/\alpha$ . Using (40, 45, 54, 89) we obtain

$$\lim_{t \rightarrow +\infty} \mathbf{J}^T \Lambda_t \mathbf{J} = 2/\tau_r, \tag{90}$$

$$\lim_{t \rightarrow +\infty} (\boldsymbol{\eta} - \tau_r \Lambda_t \mathbf{J}) = \begin{pmatrix} -1 \\ -\tau_m \vartheta \\ -1 \\ \tau_m \vartheta \end{pmatrix}, \tag{91}$$

$$\lim_{t \rightarrow +\infty} (\Lambda^{(0)} + \Lambda_t) = 2 \begin{pmatrix} 1/\tau_r & 0 & 0 & 0 \\ 0 & \tau_m & 0 & 0 \\ 0 & 0 & 1/\tau_r & 0 \\ 0 & 0 & 0 & \tau_m \end{pmatrix}. \tag{92}$$

Equations (90–92) lead to

$$\begin{aligned}
 & [(\boldsymbol{\eta} - \tau_r \Lambda_t \mathbf{J})(\boldsymbol{\eta} - \tau_r \Lambda_t \mathbf{J})^T + (t - t_0 - \tau_r^2 \mathbf{J}^T \Lambda_t \mathbf{J})(\Lambda^{(0)} + \Lambda_t)]^{-1} \\
 & \underset{t \rightarrow +\infty}{\sim} \frac{\tau_r}{2(t - t_0 - 2\tau_r) \Xi_t} \begin{pmatrix} \Xi_t - \tau_r & -\vartheta & -\tau_r & \vartheta \\ -\vartheta & \frac{\Xi_t - \tau_m \vartheta^2}{\tau_r \tau_m} & -\vartheta & \frac{\vartheta^2}{\tau_r} \\ -\tau_r & -\vartheta & \Xi_t - \tau_r & \vartheta \\ \vartheta & \frac{\vartheta^2}{\tau_r} & \vartheta & \frac{\Xi_t - \tau_m \vartheta^2}{\tau_r \tau_m} \end{pmatrix} \tag{93}
 \end{aligned}$$

with  $\Xi_t \equiv 2(t - t_0 - \tau_r + \vartheta^2 \tau_m)$ . By (53, 91, 93), we obtain

$$\Omega_t \underset{t \rightarrow +\infty}{\sim} \frac{\tau_r + \tau_m \vartheta^2}{t - t_0 - \tau_r + \tau_m \vartheta^2}. \tag{94}$$

From (55, 90, 94) we derive (56).

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